Identifying polynomials

Polynomials are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 .$$

$(a_n \neq 0)$.

The highest exponent n is called the degree.

a_n is called the leading coefficient.

Exercise. Identify polynomials.

(a) $f(x) = 3 - 2x^5$

This is a polynomial of degree 5.

(b) $f(x) = \sqrt{x} + 1$

$= x^{1/2} + 1$

This is not a polynomial. x is raised to $\frac{1}{2}$ which is not a integer.

(c) $g(x) = 2$

This is a polynomial of degree zero.

(d) $h(x) = 4x^5 (2x-3)^2$

$= 4x^5 (4x^2 - 12x + 9)$

$= 16x^7 - 48x^6 + 36x^5$

This is a polynomial of degree 7.

(e) $F(x) = \frac{1}{x} + 2$

(f) $k(x) = 3x^8 (x-2)^2 (x+1)^3$.

Degree $= 13$

No need to expand.

ANALOGY: Recall the construction of base-10 system. We write

$105 = 1\cdot 10^2 + 0\cdot 10 + 5$

$36 = 3\cdot 10 + 6$

$521 = 5\cdot 10^2 + 2\cdot 10 + 1$

$1090 = 1\cdot 10^3 + 9\cdot 10$

In general any integer N can be written as

$$N = \square\cdot 10^n + \square\cdot 10^{n-1} + \dots + \square\cdot 10^2 + \square\cdot 10 + \square$$

You are supposed to fill in the boxes with nonnegative integers, 0 is allowed.

Formally we say

$$N = a_n 10^n + a_{n-1}10^{n-1} + \dots + a_2 10^2 + a_1 10 + a_0$$

Thus N is a polynomial of 10. Note that 10 is not raised to negative exponents. If we do that we get decimals too.

[Note: The base-10 is not unique. The Babylonians used base-60].

$$f(x) = \underset{a_n}{\square}\cdot x^n \dots + \underset{a_2}{\square}\cdot x^2 + \underset{a_1}{\square}\cdot x + \underset{a_0}{\square}$$

Ques. Why is a function like $f(x) = (x-2)^5 \left(x+\frac{1}{2}\right)^{79} x^{13}$ a polynomial?

Ans.

Lemma 1 $(x-a)^n$ and $(x+a)^n$ are polynomials where a is any real number and n is a nonnegative integer.

Examples: $x-5$, $x+5$, $(x+2)^2 = x^2+2x+2$, $(x-2)^3 = x^3 - 6x^2 + 12x - 8$, etc.

Proof. $$(x-a)^n = \underbrace{(x-a)(x-a)(x-a) \cdots (x-a)}_{n \text{ times}}$$

It should be clear that if you foil it n times you will get some polynomial in x with degree n.

Lemma 2 Product of two polynomials is a polynomial

Examples

$$(x^2+2x+3)(x^{100}-5)$$
$$= x^{102} - 5x^2 + 2x^{101} - 10x + 3x^{100} - 15$$
$$= x^{102} + 2x^{101} + 3x^{100} - 5x^2 - 15$$

Proof. Let $f(x)$ and $g(x)$ be polynomials. Say that
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$
$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0.$$
Then $f(x)\cdot g(x) = (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)(b_mx^m + \cdots + b_0)$

By distributing it should be clear that we get a polynomial. Moreover the

Degree of $f \cdot g$ = Degree of f + Degree of g.

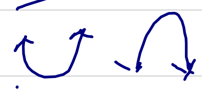
Thus,

$$f(x) = (x-2)^5 \left(x+\frac{1}{2}\right)^{79} x^{13}$$ is a polynomial

as $(x-2)^5$, $\left(x+\frac{1}{2}\right)^{79}$ and x^{13} are polynomials by lemma 1, and since the product of polynomials is a polynomial, $f(x)$ is a polynomial.

Moreover the

$$\text{Degree of } f = 5 + 79 + 13$$
$$= 97$$

Degree	Polynomial	Name	Graph
0	$f(x) = c$	constant	
1	$f(x) = mx+b$	linear	
2	$f(x) = ax^2+bx+c$	Quadratic	

Fact: Graphs of polynomials are continuous and smooth.
Continuous roughly means no gaps or holes.
Ex. These are not continuous:

Smooth roughly means no sharp corners.
Ex. These are not smooth

You need calculus to define these terms rigorously.
But all we need to know is that polynomials are very nice.

Continuous and smooth

Power functions

Power functions are functions of the form

$$f(x) = ax^n$$

where $a \neq 0$, $a \in \mathbb{R}$ and n is a positive integer (>0).

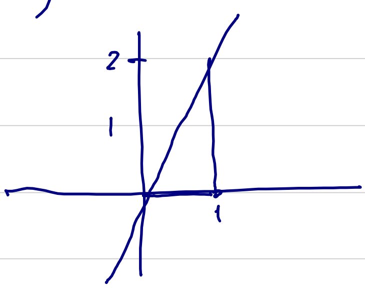
These are the building blocks of polynomials.

Case $n = 1$

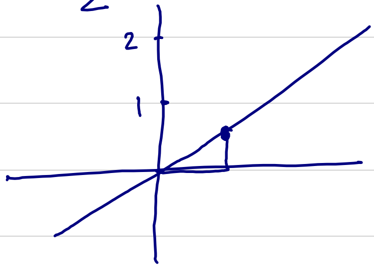
$$f(x) = ax$$

This is just a linear function with slope a. Note y-intercept is 0.

Ex. $f(x) = 2x$



$$f(x) = \frac{1}{2}x$$



Note power functions are the individual components of polynomials

Ques. Why study power functions.

Ans. The graphs of polynomials for $n \geq 3$ has a lot of information that needs to be kept track of. Instead we study power functions which have very well behaved graphs.

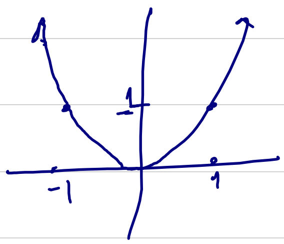
Case: $n=2$.

$$f(x) = ax^2$$

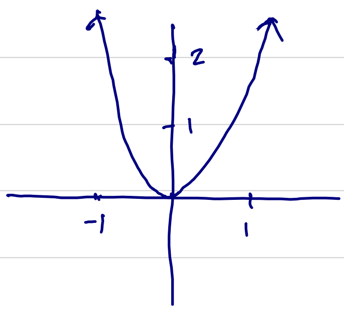
This is a quadratic function. Note $bx=0$ and $c=0$. So y-intercept $=0$. Using transformations we know that we need to stretch or compress the graph $y=x^2$ vertically by a factor of a.

<u>Examples</u>.

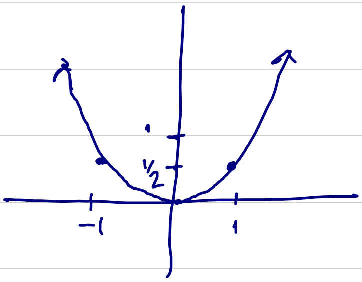
$f(x) = x^2$



$f(x) = 2x^2$



$f(x) = \frac{1}{2}x^2$



Case $n=3$:

$$f(x) = ax^3$$

This is a cubic function. What is y-intercept?

$$f(0) = a \cdot 0^3$$
$$= 0.$$

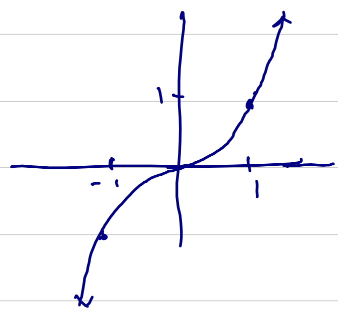
It is 0.

The graph is the graph of $y = x^3$ stretched or compressed vertically according to whether $a > 1$ or $0 < a < 1$.

Examples

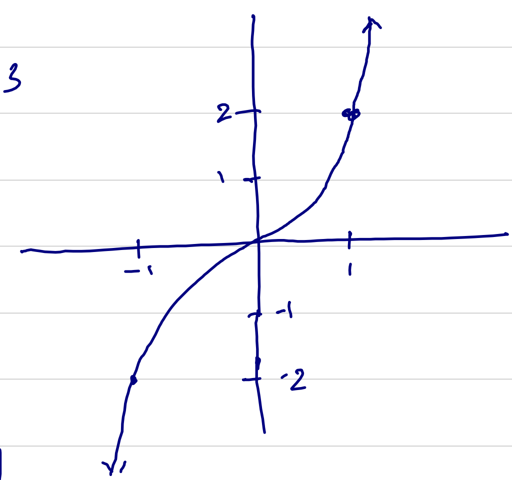
$n=3$, $a=1$

$$f(x) = x^3$$

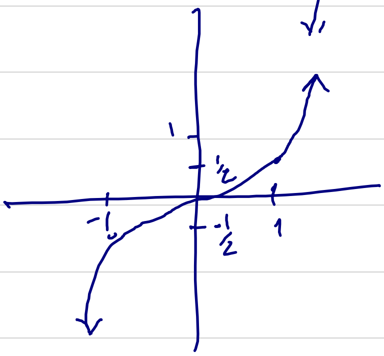


$n=3$, $a=2$

$$f(x) = 2x^3$$



$$f(x) = \frac{1}{2}x^3$$

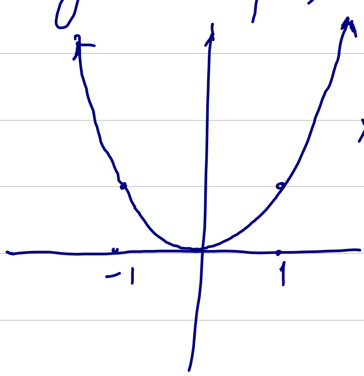


Exercise

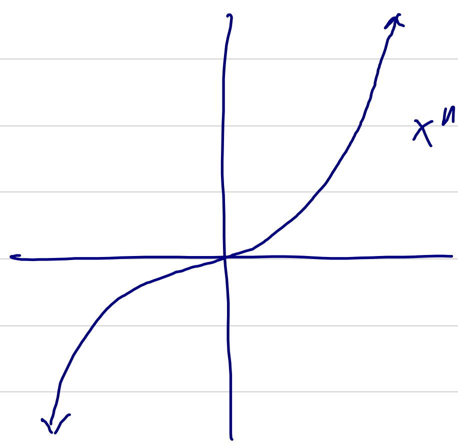
Draw the graph of $f(x) = x^4$

Draw the graph of $f(x) = x^5$

Do you notice a pattern? Consider $a = 1$ for now.
So $f(x) = x^n$. What pattern do you see?.
In general $f(x) = x^n$ looks like



for even n



for odd n.

Lets compare $f(x) = x^2$ and $f(x) = x^4$.
For $-1 < x < 1$ notice that the graph of x^4 stays below the graph of x^2.
Why?

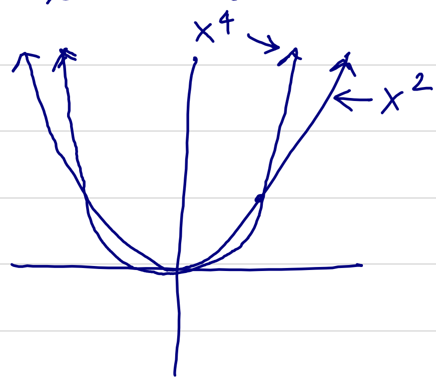
Because $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ but $\left(\frac{1}{2}\right)^4 = \frac{1}{2^4} = \frac{1}{16}$

and $\frac{1}{16} < \frac{1}{4}$.

$\left(\frac{1}{3}\right)^2 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ but $\left(\frac{1}{3}\right)^4 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{81}$

and $1/81 < 1/9$.

So as you increase the exponents, for $-1 < x < 1$, the graph shrinks below.



But the opposite happens for $x > 1$ and $x < 1$. Why.

Because when $2^2 = 4$ but $2^4 = 16$ and $16 > 4$

$3^2 = 9$ but $3^4 = 81$ and $81 > 9$.

Exercise

Compare the graphs of $f(x) = x^3$ and $f(x) = x^5$.

Now once you know the general form of $f(x) = x^n$, the graph of $f(x) = ax^n$ is obtained by stretching or compressing by a factor of a.

Analyze the following table:

	n even	n odd
Symmetry	symmetric wrt Y-axis	symmetric with respect to origin (odd function)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[0, \infty)$	$(-\infty, \infty)$
Increasing	$(0, \infty)$	$(-\infty, \infty)$
Decreasing	$(-\infty, 0)$	Nowhere

Exercise

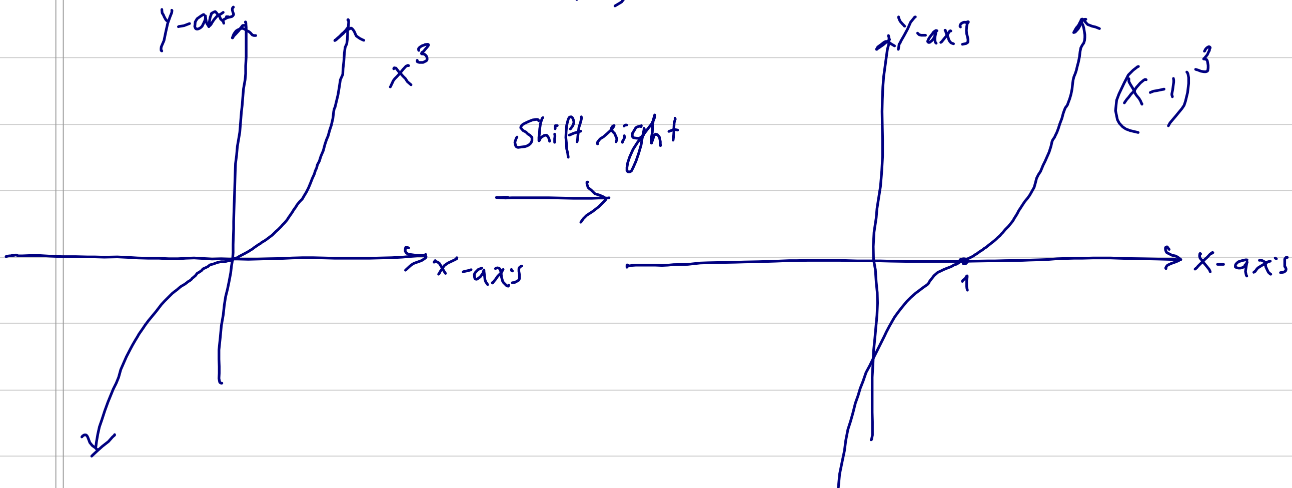
Graph $f(x) = (x-1)^5$

Soln. The main function is x^5. The transformations are:

$$M(x) \xrightarrow{\text{Shift right } 1} M(x-1)$$

$$x^5 \longmapsto (x-1)^5$$

I know x^5 roughly looks the same as x^3. It is below x^3 between $(-1, 1)$ and above x^3 elsewhere.

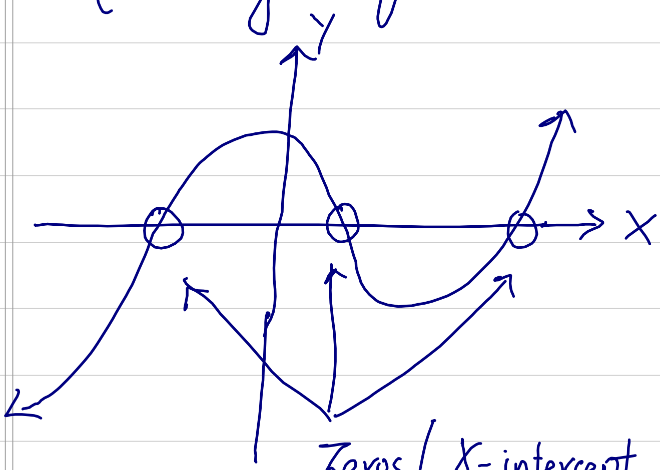


Exercise

Graph $f(x) = 1 - x^4$

"Real zeros" of a polynomial

For a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, the real zeros are all x such that $f(x) = 0$, i.e. they are the real roots/solutions of the eqn. $f(x) = 0$. Equivalently they are also the X-intercepts.

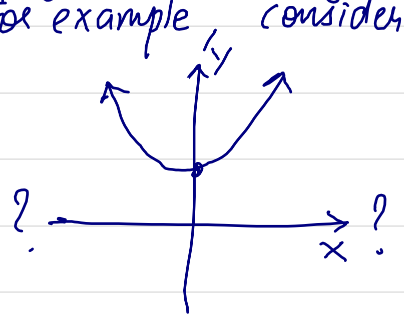


Zeros / X-intercept

Minor note: The X-intercepts are points. So they have coordinates $(x, 0)$ where x is the zero.

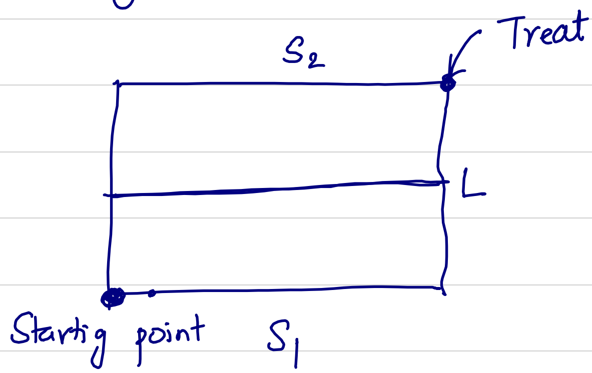
Ques. Does a polynomial always have a real zero?

Ans. No. For example, consider $f(x) = x^2 + 1$



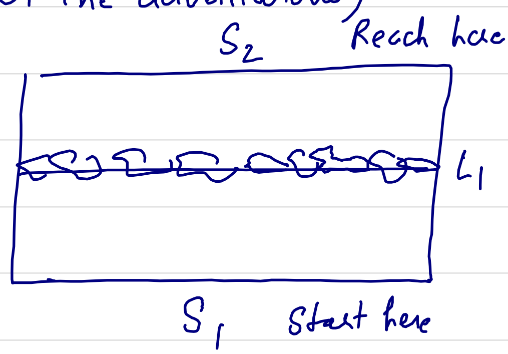
The intermediate Value Theorem is equivalent to the following two games:

Game 1



Start from the left bottom corner. Without lifting your pencil you have to draw a curve and reach the treat on the upper right hand corner. But once you touch the line segment L you loose. Is it possible to win this game?

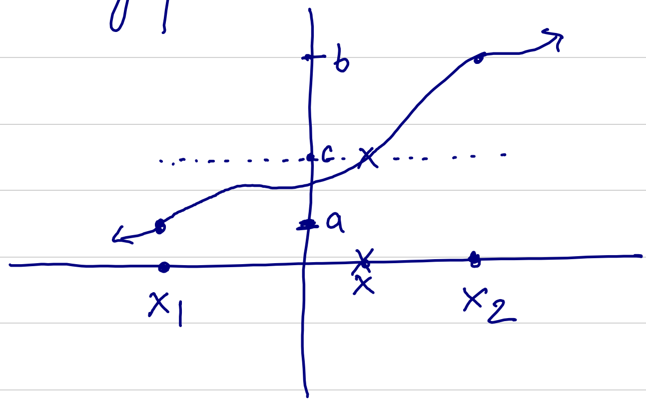
Game 2. (For the adventurous)



There is a swimming pool. You start from the bottom side. You need to swim to the other side S_2, but on the line L_1 there are a bunch of hungry crocodiles. Can you swim across safely?

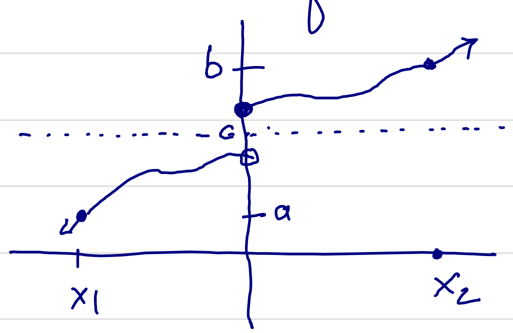
Intermediate Value Theorem

Take a continuous function, i.e a function $f(x)$ no holes or gaps.



Let $x_1, x_2 \in X$. Say that $f(x_1) = a$, $f(x_2) = b$ and $b > a$. Since f is continuous, f takes every value between a and b, i.e. if you give me any c between a and b, in other words, $c \in [a, b]$, then I can find an x between x_1 and x_2 such that $f(x) = c$.

This is a nice property continuous functions have. This is not true for all functions. Let's take a discontinuous function drawn below



This horizontal line does not intesect the graph.

So if I take a c as shown in the graph, then there is no x between x_1 and x_2 such that $f(x) = c$.

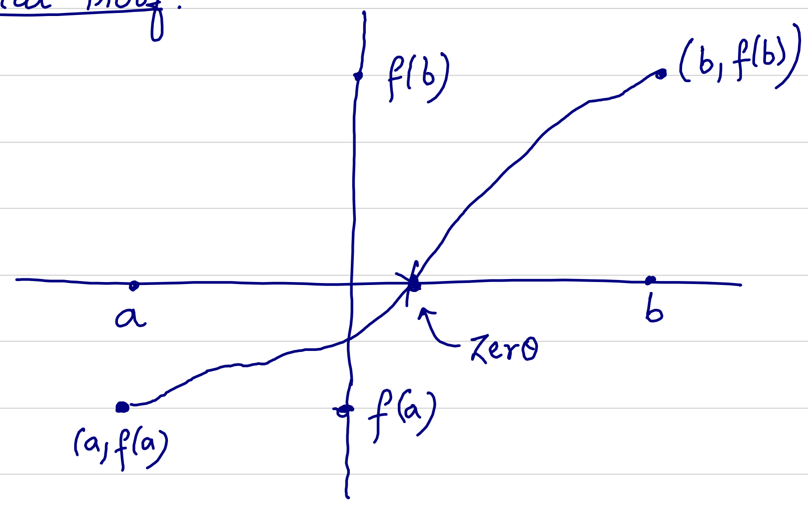
The previous observation gives us the following theorem:

<u>Intermediate Value Theorem</u>

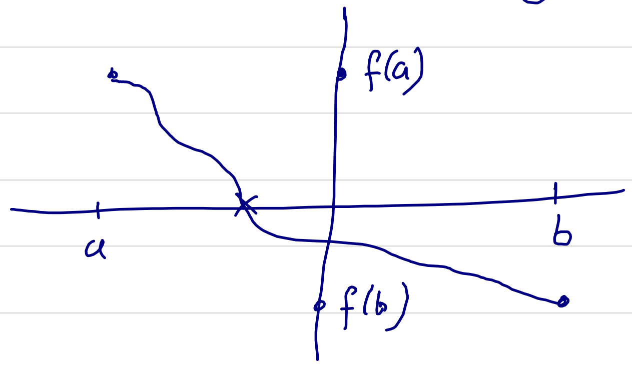
Let a and b be real numbers such that $a < b$ and let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero between a and b.

Note: A zero is a $x \in X$ such that $f(x) = 0$. It is just a X-intercept.

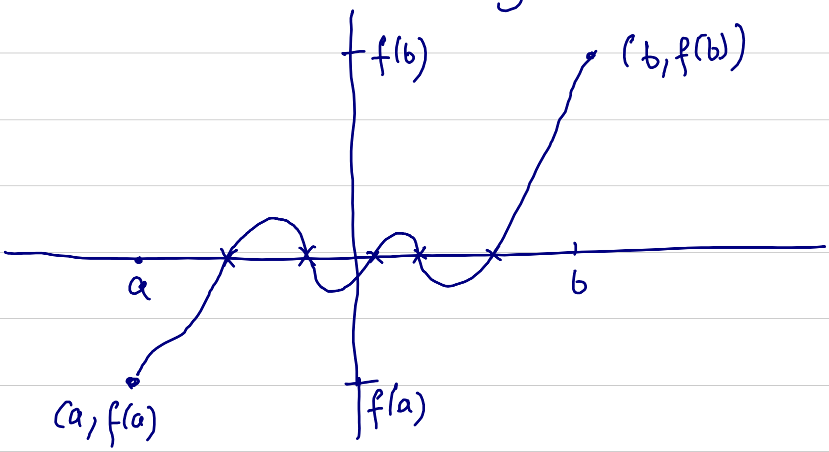
<u>Graphical Proof</u>:



Since f is a polynomial, f is continuous. Hence, the graph of f must cross the X-axis. So there must be at least one zero.



Note there could be several zeros



Ex. Find the zeros of the polynomial

$$f(x) = x^3 - 7x^2 + 12x.$$

Soln. Recall

$$\{\text{Zeros}\} = \{X\text{-intercepts}\} = \{x \text{ st. } f(x) = 0\}$$

we have to solve

$$f(x) = 0$$

$$\Rightarrow x^3 - 7x^2 + 12x = 0$$

$$\Rightarrow x(x^2 - 7x + 12) = 0$$

$$\Rightarrow x(x^2 - 4x - 3x + 12) = 0$$

$$\Rightarrow x(x(x-4) - 3(x-4)) = 0$$

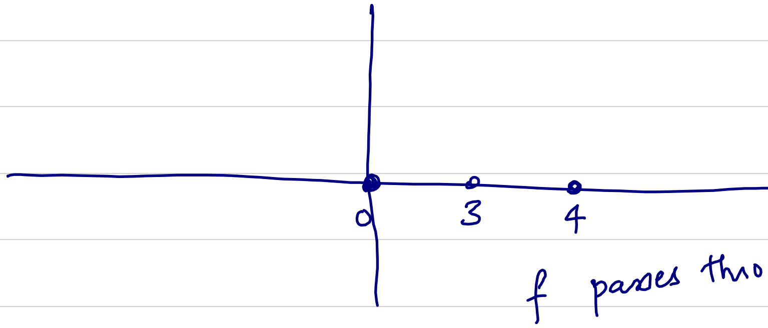
$$\Rightarrow x(x-4)(x-3) = 0$$

$\therefore$ $x = 0$ or $x = 4$ or $x = 3$

2	12
2	6
3	3
	1

$12 = (-4)(-3)$

$-7 = (-4) + (-3)$



f passes through these points

Ex.

Find zeros of $f(x) = (x-1)^2$.

Soln. We have to solve

$$f(x) = 0$$

i.e. $$(x-1)^2 = 0$$

$$\Rightarrow \quad (x-1)(x-1) = 0$$

Thus, $x = 1$ or $x = 1$.

But they are the same,

Ques Why write it twice?

Ans. It is important to keep track of the number of times a zero appears in a polynomial.

2 Reasons:

1) It gives us information about the graph of the polynomial near the zero

2) We have to know whether we have found all the zeros or not.

Ex. $(x-1)^2$ is a quadratic, so there are at most two zeros. We know 1 is a zero with multiplicity 2 so there cannot be any other solution.

We will count the solutions with multiplicity. And the total number of zeros with multiplicity must equal n (the degree of polynomial).

We say that $x = 1$ is a repeated root/solution with multiplicity 2.

If $(x-a)^n$ is a factor of a polynomial f, then a is a <u>zero of multiplicity n</u> of f

<u>Ex.</u> $g(x) = (x-1)^2 \left(x+\frac{3}{5}\right)^7 (x+5)$

<u>Sol^n</u> The degree is $2+7+1=10$

The zeros / roots are

1 with multiplicity 2

<u>Finding a polynomial from its zeros</u>

Find a polynomial of degree 7 whose zeros are

-2 (multiplicity 2) 0 (multiplicity 4) 1 (multiplicity 1)

If $x=a$ is a zero then $(x-a)$ is a factor. So

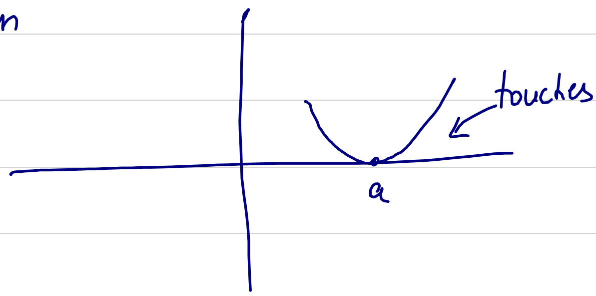
$$\begin{aligned} f(x) &= (x+2)^2 (x-0)^4 (x-1) \\ &= (x+2)^2 x^4 (x-1) \\ &= (x^2+2x+4)\, x^4 (x-1) \\ &= x^4(x^3+3x^2-4) \\ &= x^7+3x^6-4x^4 \end{aligned}$$

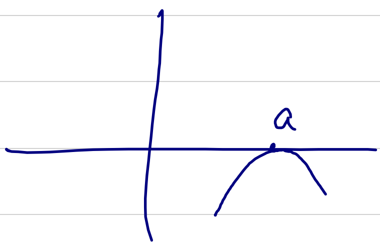
This example shows how easy it is to find the zeros of the polynomial if you know the factored form. That's why factoring is important.

Relation of Multiplicity of a zero to the graph

Let $x = a$ be a zero of multiplicity n. we have 2 cases:

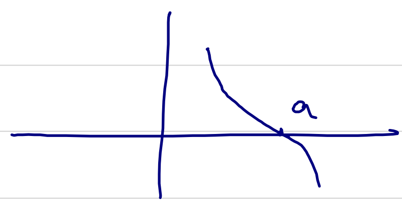
1) n even





In this case the graph touches the X-axis.

2) n odd



In this case the graph crosses the X-axis

Exercise

Why?

Theorem on End Behavior

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

The graph of f has the same end behavior as the power function $y = a_n x^n$.

$\left\{\begin{array}{l}\text{Note: End behavior means the behavior of the} \\ \text{graph as } x \to \infty \text{ and } x \to -\infty.\end{array}\right.$

(Not Required) Proof. We can write f as follows

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \cdot \frac{1}{x} + \dots + \frac{a_2}{a_n} \cdot \frac{1}{x^{n+2}} + a_1 \cdot \frac{1}{x^{n+1}} + \frac{a_0}{x^n}\right)$$

Note that as $x \to \infty$,

$$\frac{1}{x} \to 0, \quad \frac{1}{x^2} \to 0 \quad \& \dots \quad \frac{1}{x^n} \to 0$$

Therefore as $x \to \infty$,

$f(x)$ is almost the same as $a_n x^n \cdot (1 + 0 + 0 \dots 0)$

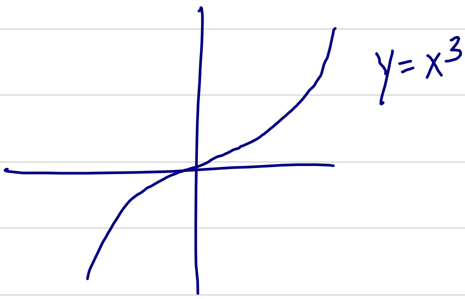
$$= a_n x^n \cdot 1$$

$$= a_n x^n.$$

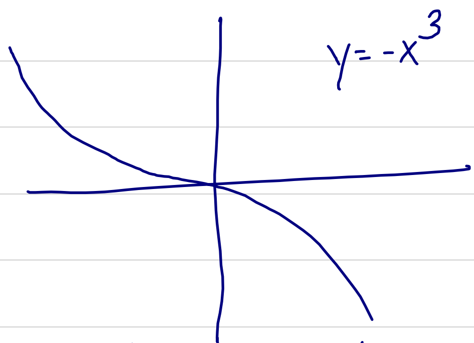
But we already know the graphs of power functions and their end behaviors.

Mnemonic

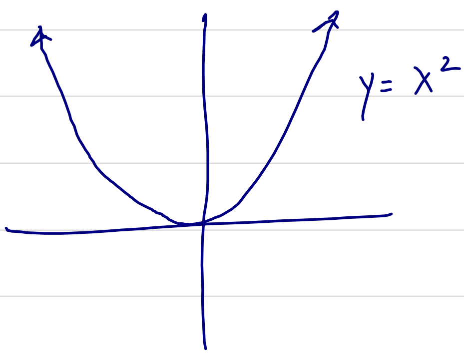
Just Remember 4 cases



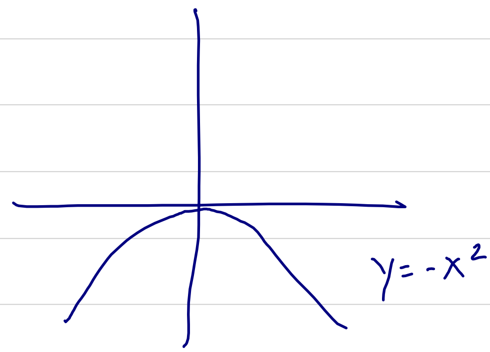
n odd a_n positive



n odd a_n negative



n even a_n positive



n even a_n negative

Graphing a polynomial

Ingredients:

1) Y-intercept
2) X-intercepts or zeros
3) Multiplicities of zeros
4) End behavior
5) Some additional test points.

Sketch the graph of $f(x) = x^5 - 4x^3$

Solution

Step 1 (Y-intercept)

$$f(0) = 0^5 - 4 \cdot 0^3$$
$$= 0.$$

So $(0,0)$ is a point on the graph.

Step 2 (X-intercepts)

We must solve

$$f(x) = 0$$
$$x^5 - 4x^3 = 0$$
$$x^3(x^2 - 4) = 0$$

Either $x^3 = 0 \Rightarrow x = 0$

or, $x^2 - 4 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2.$

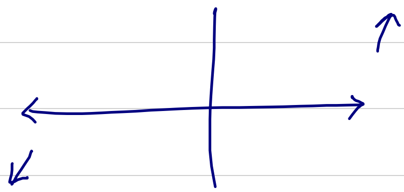
$\therefore x = 0, 2, -2$ are zeros.

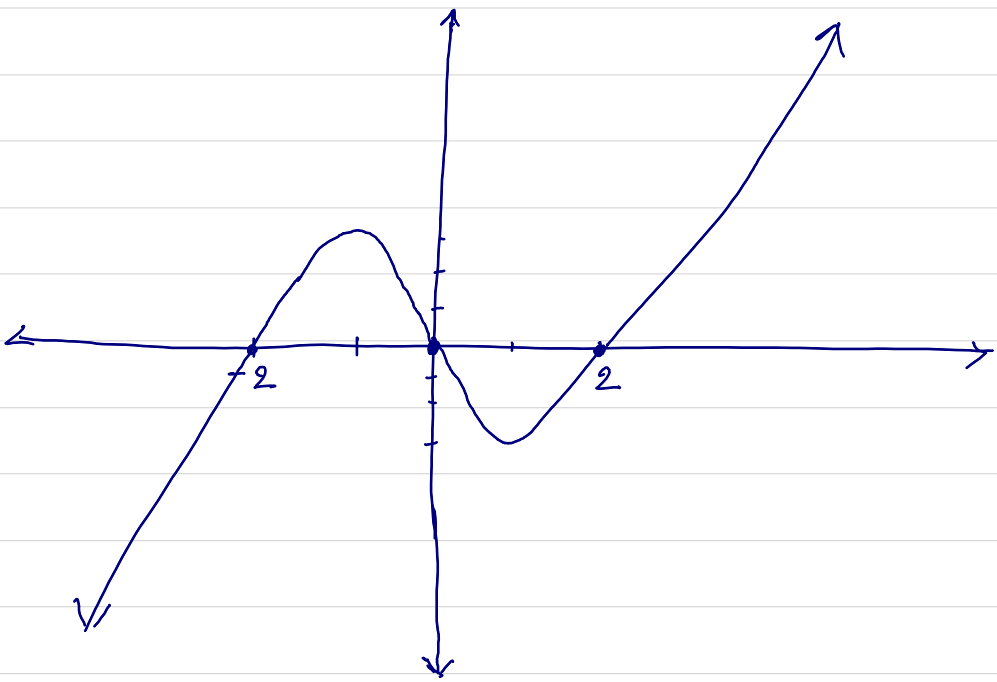
0	2	-2
multiplicity 1	mult. 1	multiplicity 1

Step 3 (End Behavior)

$f(x) = x^5 - 4x^3$ behaves like x^5 at the ends.

so





Step 4 Additional points

Example Plug 1

$f(1) = 1^5 - 4 \cdot 1^3$

$= 1 - 4$

$= -3$

$(x-1)^2 (x+3) (x+1)^2$

$(x^2-2x+1)(x^2+2x+1)(x+3)$ Plug -1

$f(-1) = (-1)^5 - 4(-1)^3$

$= -1 + 4$

$= 3$

$$f(x) = x^5 + 3x^4 - 2x^3 - 6x^2 + x + 3$$
$$= x^4(x+3) - 2x^2(x+3) + (x+3)$$
$$= (x+3)(x^4 - 2x^2 + 1)$$
$$= (x+3)\left((x^2)^2 - 2\cdot x^2 + 1^2\right)$$
$$= (x+3)(x^2-1)^2$$
$$= (x+3)(x+1)^2(x-1)^2$$